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An exact solution for Parisi equations with R steps of RSB, Free energy and fluctuations

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Abstract

We show that there is no need to modify the Parisi replica symmetry breaking ansatz, by working with R steps of breaking and solving *exactly* the discrete stationarity equations generated by the standard 'truncated Hamiltonian' of spin glass theory.

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1. Introduction

In a quite recent work Aspelmeier and Moore [1] (hereafter referred to as AM) have considered the sample-to-sample free energy fluctuations in finite dimensional spin glasses via the replica method. To that effect they reconsider higher order terms in the replica number n and they conclude that the Parisi symmetry breaking scheme [2] does not give the correct answer for these higher order terms. Finally, they propose a modified symmetry breaking scheme that resolves the problem.

What we set out to do here is as follows. Starting from the same truncated Hamiltonian (AM.3) *we solve exactly the discrete stationarity equations* for R steps of replica symmetry breaking, namely we obtain the $R+1$ values of $q_{\alpha\beta}$ indexed by their overlap values $q_0, q_1, \ldots, q_R$ (together with $q_{\alpha\alpha} \equiv q_{R+1} = 0$) and the R values of Parisi box sizes $p_1, p_2, \ldots, p_R$ together with the two *fixed* boundary values $p_0 = n$ and $P_{R+1} = 1$. As a result, we find two families (a), (b) of solutions associated with two possible values of q_0, namely, letting $g = w/(2y)$:

(a) $q_0 = \frac{3n}{2}g$. In this case the corresponding free energy is identical to the Kondor [3] result

$$nf^{(a)}(n) = nf - \frac{9n^6}{640}wg^3. \qquad (1.1)$$

(b) $q_0 = 0$. The free energy is now larger

$$nf^{(b)}(n) = nf. \tag{1.2}$$

Solution (b) is therefore the appropriate one to choose[1], both solutions having a non-negative Hessian spectrum when $R \to \infty$. Among the family of solutions (b) with $q_0 = 0$ and free energy $f^{(b)}$, we will pick a *reference* solution with a set of values $q_t, p_t, t = 1, 2, \ldots, R$. All the other solutions will be shown elsewhere [5] to correspond to a (discrete) reparametrization for large R. With that set of values, we proceed and compute the contribution to fluctuations, with a result that matches for $R \to \infty$ the Aspelmeier and Moore ones [1]. We thereby establish that there is indeed no need to modify the Parisi replica symmetry breaking scheme.

2. Solution of the stationarity equations

The stationarity equations are derived from the free energy functional

$$nf = -\sum_{t=0}^{R+1} \left\{ (p_t - p_{t+1}) \left[\frac{\tau}{2} q_t^2 + \frac{y}{12} q_t^4 \right] + \left(\frac{1}{p_t} - \frac{1}{p_{t-1}} \right) \frac{w}{6} \hat{q}_t^3 \right\} \tag{2.1}$$

where we have used the replica Fourier transform $\hat{q}$ of q [6][2]

$$\hat{q}_k = \sum_{t=k}^{R+1} p_t (q_t - q_{t-1}) = \sum_{t=k}^{R} p_t (q_t - q_{t-1}) - q_R. \tag{2.2}$$

Combining the stationarity equations, we obtain in the end

$$\begin{aligned} &g p_t = \tfrac{1}{2}(q_t + q_{t-1}) \qquad t = 1, 2, \ldots, R \\ &(q_t - q_{t-1})^2 = (q_{t-1} - q_{t-2})^2 = \cdots = (q_1 - q_0)^2 \qquad t = 1, 2, \ldots, R. \end{aligned} \tag{2.3}$$

Here we concentrate on the particular *reference* solution such that

$$q_t - q_{t-1} = q_{t-1} - q_{t-2} = \cdots = q_1 - q_0 = \frac{q_R - q_0}{R} \tag{2.4}$$

which leads to

$$\begin{aligned} q_t &= q_0 + (q_R - q_0)\frac{t}{R} \qquad t = 0, 1, \ldots, R \\ g p_t &= q_0 + (q_R - q_0)\frac{2t-1}{2R} \qquad t = 1, 2, \ldots, R \end{aligned} \tag{2.5}$$

together with $q_{R+1} = 0$, $p_{R+1} = 1$. Besides one has two more equations that determine q_0 and q_R:

$$E(q_R) - \frac{y}{6}\left(\frac{q_R - q_0}{R}\right)^2 = 0 \tag{2.6}$$

where $E(q_R) = \tau - w q_R + y q_R^2$, and which is valid for $R > 0$, and

$$q_0 \left(E(q_R) + y \frac{q_0}{3}(3 g p_0 - 2 q_0) \right) = 0 \tag{2.7}$$

valid for all R. Note that if $R = 0$, $q_R \equiv q_0$, then (2.5) is a tautology and only (2.7) survives, leading to the standard result $wq = 2\tau/(2 - p_0) + O(\tau^2)$. In fact, one is interested in the limit of large R, whereby (2.6) yields the relationship

$$E(q_R) = 0 \tag{2.8}$$

[1] Assuming that the proof given by Guerra [4] extends to small finite values of n.

[2] In the continuum limit, the transform was first used by Mezard and Parisi, see [6].

and from (2.7) either $q_0 = 0$ or $q_0 = 3gp_0/2$ as, respectively, in the cases (b) and (a). Note that q_t is monotonic except for its last step ($q_{R+1} = 0$), and p_t is monotonic except for its first step (when p_0 is kept fixed at a value $n \neq 0$).

In the *continuum limit*, where $t/R \to x$ and $q_t \to q(x)$, $p_t \to p(x)$, we get for x in the *open* interval (0, 1)

$$q(x) = gp(x) = q_R x \qquad 0 < x < 1. \tag{2.9}$$

We now proceed to get the fluctuation contribution as in (AM.5).

3. Fluctuations: the replicon sector

We have as in (AM.9)

$$n\delta f_{\text{rep}} = \frac{V}{2}\int \frac{\mathrm{d}^D p}{(2\pi)^D} I_{\text{rep}}(p) \tag{3.1}$$

where

$$I_{\text{rep}}(p) = n\sum_{r=0}^{R}\sum_{k,l=r+1}^{R+1} \mu(r;k,l)\log(p^2 + \lambda(r;k,l)). \tag{3.2}$$

Here the replicon eigenvalue λ is

$$\lambda(r;k,l) = -2\tau - w\hat{q}_k - w\hat{q}_l - 2yq_r^2. \tag{3.3}$$

The multiplicity $\mu(r;k,l)$ [7][3] is given by

$$\mu(r;k,l) = \tfrac{1}{2}(p_r - p_{r+1})\mu(k)\mu(l) \tag{3.4}$$

where

$$\mu(k) = \begin{cases} \frac{1}{p_k} - \frac{1}{p_{k-1}} & k > r+1 \\ \frac{1}{p_{r+1}} & k = r+1. \end{cases} \tag{3.5}$$

We note that p_0 is absent from $\hat{q}$, since even if the indices k, l were allowed to take the value 0, it would appear in the vanishing combination $p_0 q_0$. The p_0 dependence can therefore only arise from the multiplicity. Collecting the terms in p_0 we get

$$I_{\text{rep}} = \frac{n}{2}p_0\sum_{k,l=1}^{R+1}\mu(k)\mu(l)\log(p^2 + \lambda(0;k,l)) + \frac{n}{2}\sum_{r=1}^{R} p_r \sum_{k,l=r+1}^{R}\log\left(\frac{p^2 + \lambda(r;k,l)}{p^2 + \lambda(r-1;k,l)}\right). \tag{3.6}$$

Using (2.5)–(2.7) and (3.1)–(3.5) we get

$$\begin{aligned}\lambda(r;k,l) &= -2E(q_R) + 2y\left(\frac{q_R}{R}\right)^2\left(\frac{1}{2}(k^2 + l^2) - r^2 - (k+l) + 1\right) \\ &= 2yq_R^2\left(\frac{1}{2}\left(\left(\frac{k}{R}\right)^2 + \left(\frac{l}{R}\right)^2\right) - \left(\frac{r}{R}\right)^2\right) - 2y\left(\frac{q_R}{R}\right)^2\left(k + l - \frac{5}{6}\right).\end{aligned} \tag{3.7}$$

We note that the lowest (replicon) eigenvalue is given by

$$\lambda(r;r+1,r+1) = -\frac{y}{3}\left(\frac{q_R}{R}\right)^2 \qquad r = 0, 1, 2, \ldots, R \tag{3.8}$$

[3] We keep here the same notation except for multiplicities in which we exhibit their factor n.

hence we find an instability, except in the limit $R \to \infty$ where this is suppressed. All other eigenvalues are positive. We thus have in the Parisi limit $R+1$ zero modes arising from the negative eigenvalues (3.8). In that limit, one has

$$I_{\text{rep}} = \frac{n}{2}\int_0^1 x\,\mathrm{d}x\,\partial_x\left\{\int_x^1 \frac{\mathrm{d}k}{k}\partial_k\int_x^1\frac{\mathrm{d}l}{l}\partial_l \log\left(p^2+yq_R^2(k^2+l^2-2x^2)\right)\right\} + \frac{n^2}{2}\int_0^1\frac{\mathrm{d}k}{k}\partial_k\int_0^1\frac{\mathrm{d}l}{l}\partial_l\log\left(p^2+yq_R^2(k^2+l^2)\right) \tag{3.9}$$

which coincides with (AM.12).

4. Fluctuations: the longitudinal-anomalous (LA) sector

We now have

$$\begin{aligned} n\delta f_{\text{LA}} &= \frac{V}{2}\int\frac{\mathrm{d}^D p}{(2\pi)^D}\delta I_{\text{LA}}(p)\\ \delta I_{\text{LA}}(p) &= n\sum_{k=0}^{R+1}\mu(k)\log\det\Delta_k(r,s)\\ \Delta_k(r,s) &= \delta^{\text{Kr}}_{r,s} - \frac{wq_{\min(r,s)}}{\Lambda_k^{(r)}}\delta_s^{(k-1)} \end{aligned} \tag{4.1}$$

where δ^{Kr} denotes the Kronecker delta, while we have

$$\begin{aligned} \delta_s^{(k-1)} &\equiv p_s^{(k-1)} - p_{s+1}^{(k-1)}\\ p_s^{(k-1)} &= \begin{cases} p_s & s\geqslant k-1\\ 2p_s & s<k-1\end{cases}\\ \Lambda_k^{(r)} &= \begin{cases} p^2+\lambda(r;r+1,k) & k\geqslant r+1\\ p^2+\lambda(r;r+1,r+1) & k<r+1.\end{cases} \end{aligned} \tag{4.2}$$

Expanding the determinant $\Delta_k(r,s)$ yields

$$\det\Delta_k(r,s) = 1+\sum_{m=1}^{\infty}(-w)^m\sum_{0\leqslant s_1<s_2<\cdots<s_m}\prod_{i=1}^{m}\left(q_{s_i}-q_{s_{i-1}}\right)\frac{\delta_{s_i}^{(k-1)}}{\Lambda_k^{(s_i)}} \tag{4.3}$$

where we have set $s_0 \equiv 0$. In order to have p_0 occurring in the determinant, i.e. in one of the $\delta_{s_i}^{(k-1)}$, we need $s_i = 0$ hence the only possible term is $s_1 = 0$, but the prefactor $q_{s_1} - q_{s_0} \equiv q_{s_1} = q_0$ vanishes and there is no p_0 contribution from the log again. The only p_0 contribution comes from the multiplicity $n\mu(k)$ which now cannot sustain an n^2 contribution[4].

5. Conclusion

With no contribution to fluctuations from the LA sector, we conclude that the full answer is given by the contribution from the replicon sector as of (3.1) and (3.9), thus corroborating the result of [1].

[4] One may then ask what becomes of the term $k = 0$ which has a factor n/p_0. It is actually given by $\delta I^0_{\text{LA}} = \log\det(\Delta_0(r,s)/\Delta_1(r,s))$, but with $\delta_s^{(0)} = \delta_s^{(-1)} - p_0\delta^{\text{Kr}}_{s,0}$, whereas $\Lambda_1(s) = \Lambda_0(s)$. Again, δI^0_{LA} reduces to a contribution $\sim p_0\delta^{Kr}_{s,0}$ which vanishes with q_0.

6. Extension to Aspelmeier, Moore, Young calculation

In [8, 9] the authors give an analytic answer to a long-standing problem: computing the interface free energy of the Ising spin glass. They show that (part of) the answer is obtained by computing $\overline{Z^n Z^m}$ and, in the associated free energy, leaving aside the terms in $(n+m)$, $(n+m)^2$ and keeping the terms in nm. From (3.6) we clearly see that, leaving aside an $(n+m)^2$ contribution, one is left with a term

$$I_{\mathrm{P}}(p) \equiv I_{\mathrm{rep}}(p) = -nm \sum_{k,l\geqslant 1} \mu(k)\mu(l) \log(p^2 + \lambda(0;k,l)) \tag{6.1}$$

replacing the quadratic term of (3.6) and associated with periodic boundary conditions. The fluctuation contributions from the off-diagonal blocks of the Hessian (mixed sector, associated with antiperiodic boundary conditions) are much easier to deal with since they are *exactly* given by

$$I_{\mathrm{AP}}(p) = +nm \sum_{k,l=0}^{R+1} \mu(k)\mu(l) \log(p^2 + \lambda(0;k,l)), \tag{6.2}$$

eigenvalues and multiplicities matching the $R = 0$ calculation presented earlier by the same authors [8]. The LA sector is represented here, respectively, by $k = l = 0$ and $k = 0, l \geqslant 1$ or $k \geqslant 1, l = 0$, the replicon sector by $k, l \geqslant 1$ as in (6.1). But with $\hat{q}_1 = \hat{q}_0$ we now see that the two contributions periodic (6.1) and antiperiodic (6.2) are formally identical but for their sign. This result again matches that proposed in [9].

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References

[1] Aspelmeier T and Moore M A 2002 Free energy fluctuations in Ising spin glasses *Preprint* cond-mat/0211707
[2] Parisi G 1979 *Phys. Lett.* A **73** 203
Parisi G 1980 *J. Phys. A: Math. Gen.* **13** L115, 1101
[3] Kondor I 1983 *J. Phys. A: Math. Gen.* **16** L127
[4] Guerra F 2002 Broken replica symmetry bounds in the mean field spin glass model *Preprint* cond-mat/0205123
[5] De Dominicis C and Di Francesco P in preparation
[6] De Dominicis C, Carlucci D M and Temesvari T 1997 *J. Physique* I **7** 105
Mezard M and Parisi G 1991 *J. Physique* I **1** 809
[7] De Dominicis C, Kondor I and Temesvari T 1997 *Spin Glasses and Random Fields* ed A P Young (Singapore/London: World Scientific)
[8] Aspelmeier T, Moore M A and Young A P 2002 Interface energies in Ising spin glasses *Preprint* cond-mat/0209290 (v1)
[9] Aspelmeier T, Moore M A and Young A P 2002 Interface energies in Ising spin glasses *Preprint* cond-mat/0209290 (v2)